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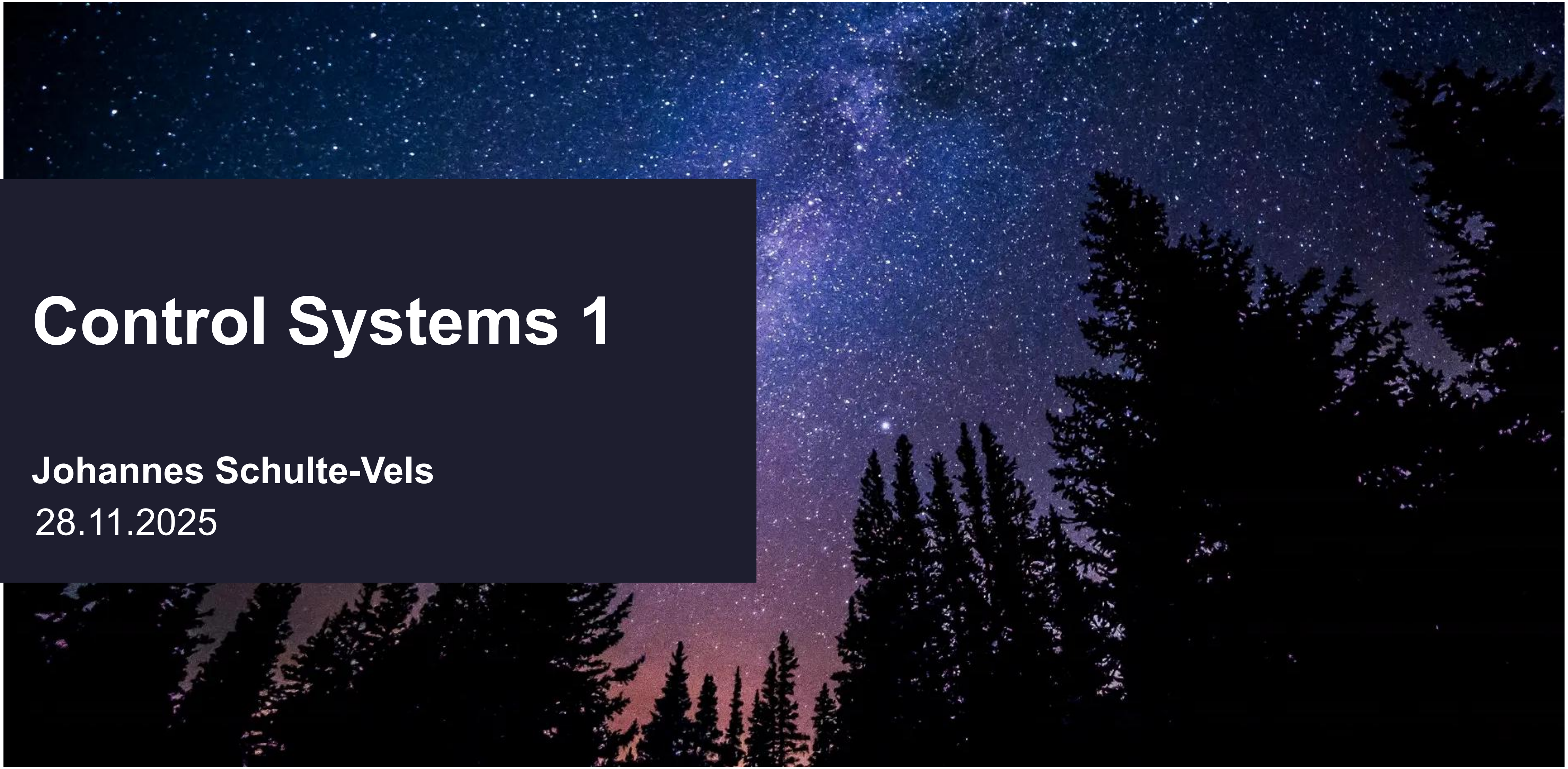
Institute for Dynamic Systems and Control



Control Systems 1

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28.11.2025

Welcome!

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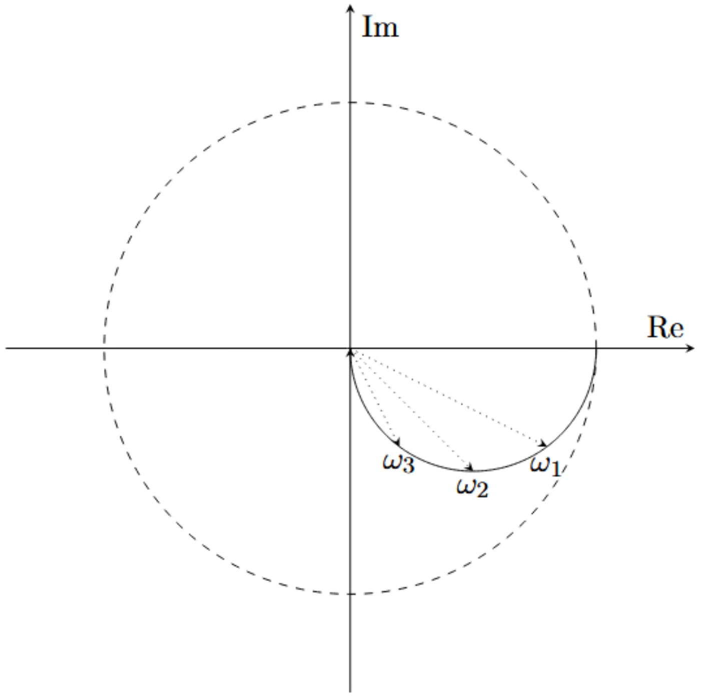
Today

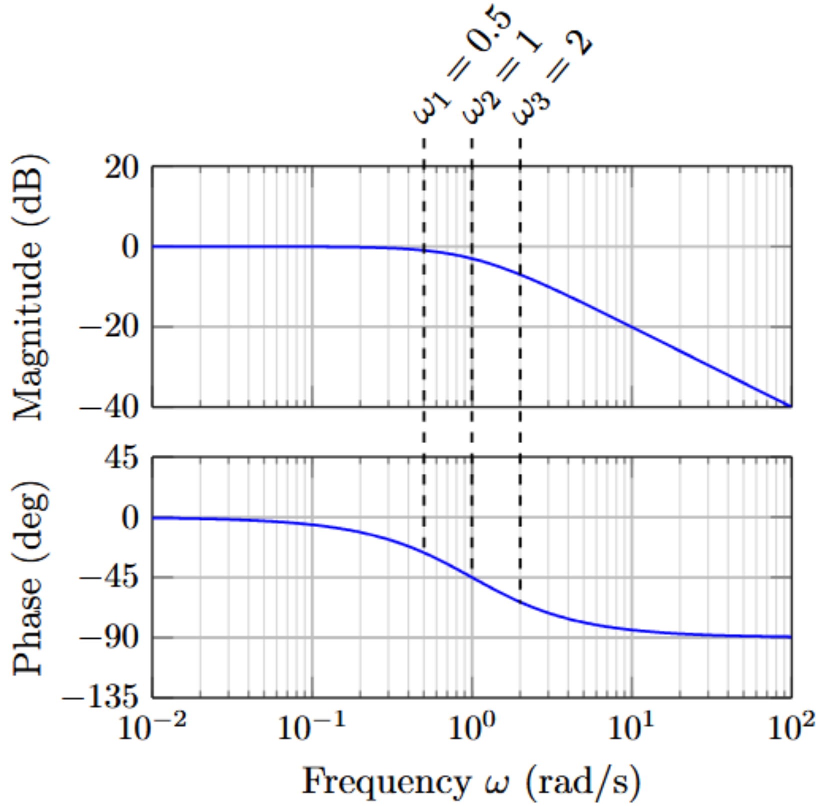
- Repetition Session 10
- Theory Recap
  - Frequency Domain Specifications
  - Bode Obstacle Course
  - c Loop shaping
- Q&A Session / Done

Repetition Session 10

Polar Plot – Bode Plot

A single parametric curve in the complex plane with ω being implicit. Every point on the curve has a certain magnitude and a certain angle (phase), meaning that for every specific ω we get the magnitude and phase of our TF.



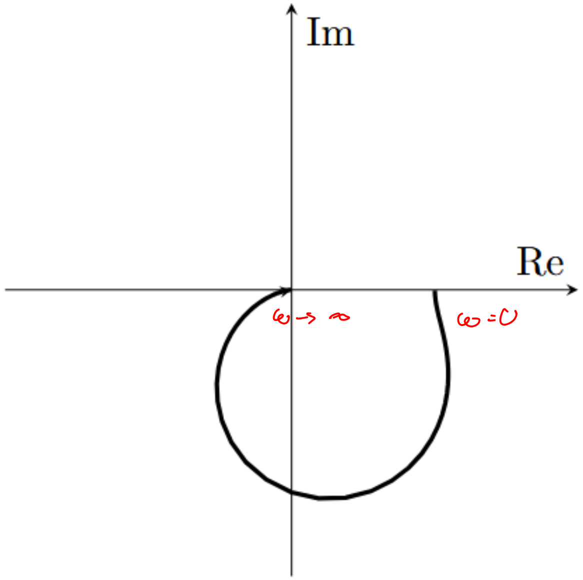


Nyquist Plot Construction

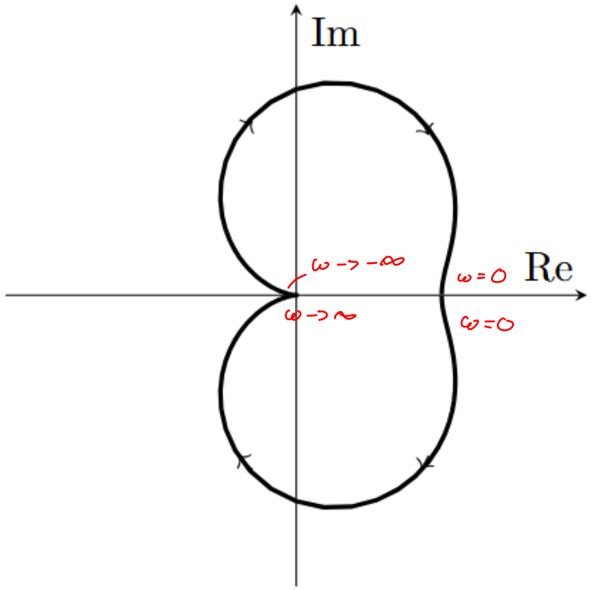
The **Polar plot** is a parametic curve in the complex plane for $\omega\colon\ 0 \rightarrow \infty$
The **Nyquist plot** is nothing but the completion of the Idea, meaning $\omega\colon -\infty \rightarrow \infty$

Simplistically stated, it is just the Polar plot mirrored on the real axis, **but continuing the direction of the parametric curve.**

Polar Plot

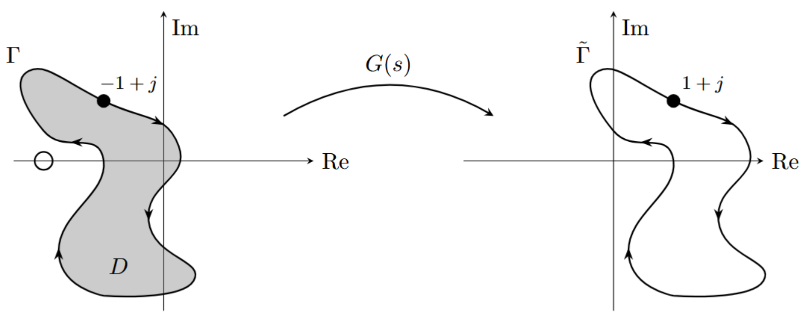


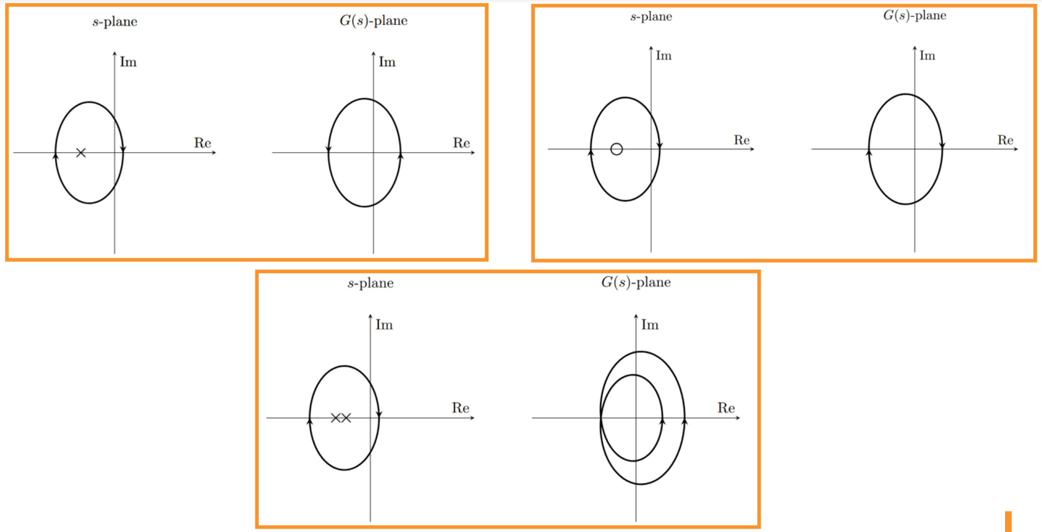
Nyquist Plot



Principle of the Variation of the Argument

$G(s) = s + 2$





Theorem (Variation of the argument [Proof in A&M, pp. 277–278])

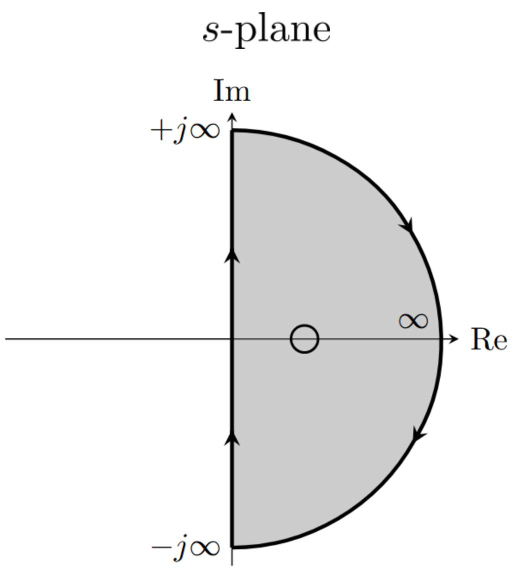
The number N of times that G(s) encircles the origin of the complex plane as s moves along the boundary Γ of a bounded simply-connected region of the plane satisfies

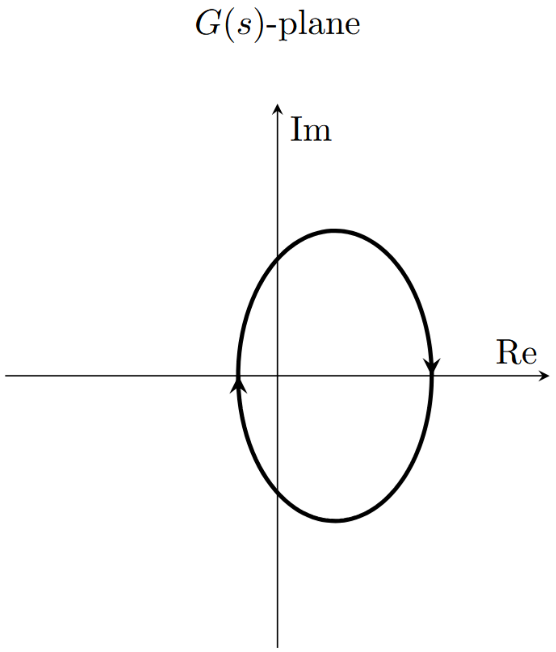
$$N = Z - P,$$

where Z and P are the numbers of zeros and poles of G(s) in D, respectively. Note that the encirclements are counted positive if in the same direction as s moves along Γ, and negative otherwise.

Nyquist Idea

Then we had the Idea to introduce a contour that includes the entire RHP and applying a TF to it. We called it the **Nyquist contour**, and it's mapping the **Nyquist plot**





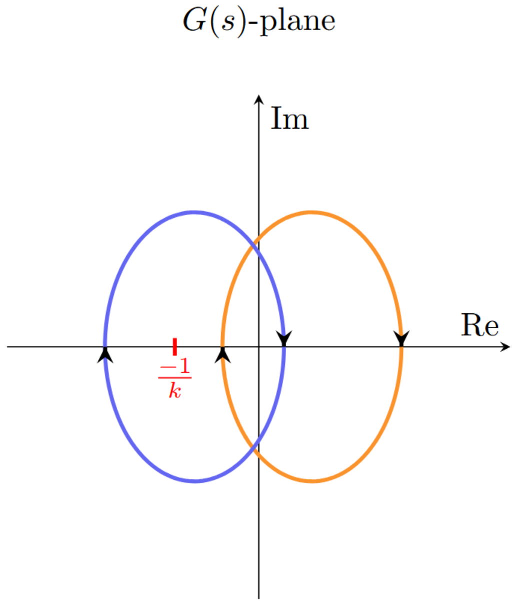
Ultimate Goal

Generally, the Idea was always to assess the closed-loop stability, by checking for the RHP poles of $T(s) = \frac{kL(s)}{1+kL(s)}$.

With help of previous theorem, the Nyquist plot, and some reforumlations, we found a very interesting formula in the end:

Enc. of $\frac{-1}{k}$ of $L(s)$ = # RHP zeros of $1 + kL(s)$ − RHP poles of $L(s)$

With $L(s)$ being the open loop TF



Derivation

For you to look at if you want. Won't go through it now. We did it last week

Enc. of origin (of Nyquist plot) of $1 + kL(s)$ = # RHP zeros of $1 + kL(s)$ − RHP poles of $1 + kL(s)$

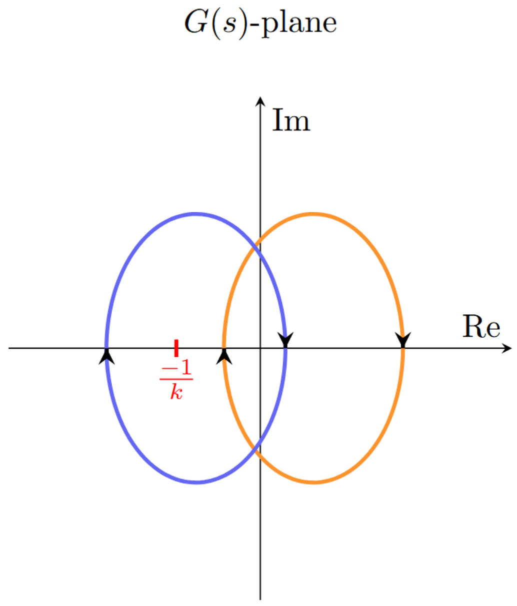
Enc. of origin of $1 + kL(s)$ = # Enc. of $\frac{-1}{k}$ of $L(s)$

Lets quickly look at the poles of $1 + kL(s)$:

Remember: $L(s) = \frac{N(s)}{D(s)}$, $1 + kL(s) = 1 + \frac{kN(s)}{D(s)} = \frac{D(s) + kN(s)}{D(s)}$

→ We can see that **they both share the same poles**, that are equal to **the open loop unstable poles**!

Enc. of $\frac{-1}{k}$ of $L(s)$ = # RHP zeros of $1 + kL(s)$ − RHP poles of $L(s)$



All coming together!!

Enc. of $\frac{-1}{k}$ of $L(s)$ = # RHP zeros of $1 + kL(s)$ − RHP poles of $L(s)$

Poles of the closed loop TF $T(s) = \frac{kL(s)}{1+kL(s)}$ = Zeros of $1 + kL(s)$.

This all leads us to the following formula N = Z – P, or reformulated:

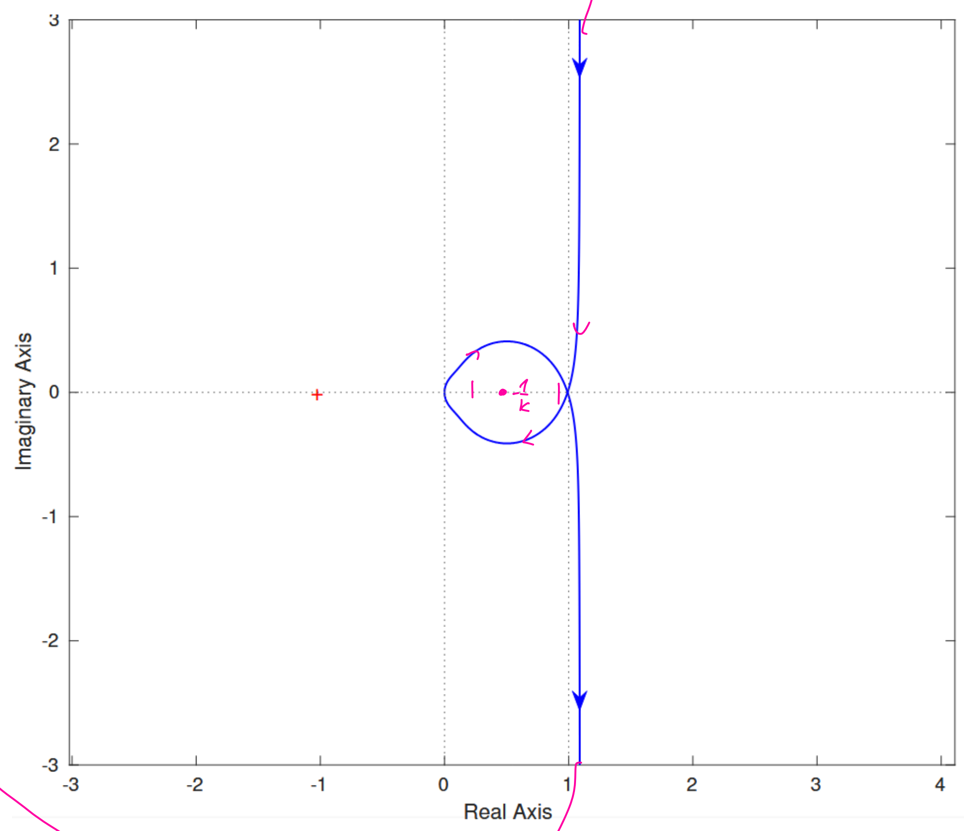
Z = N + P

Z = # Unstable Closed-Loop Poles
N = # CW enc. of $L(s)$
P = # Unstable Open-Loop Poles

This is awesome because it allows us to assess the closed loop stability by only looking at the open loop system $L(s)$

FS 2024

$$L(s) = \frac{2(s+5)(s-0.1)}{s(s+1)(s+10)}$$



What is the number of unstable poles Z of the CL system for 0.25 < − 1/k < 0.75?

A) $Z = 0$

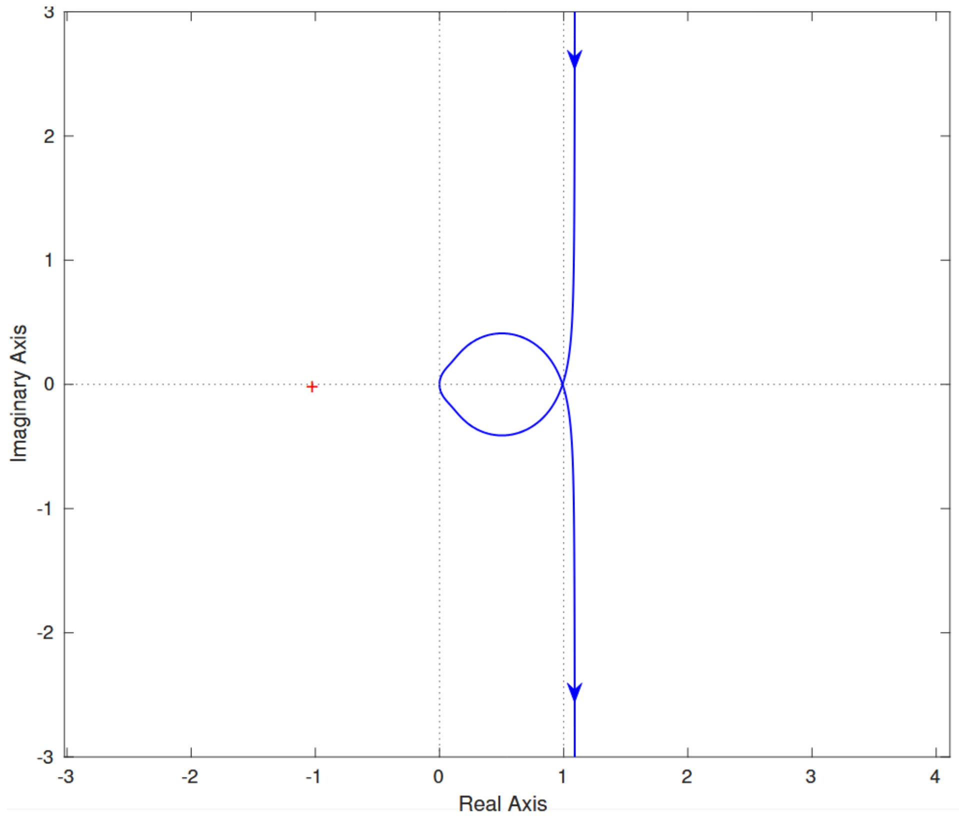
B) $Z = 1$

C) $Z = 2$

D) $Z = 3$

FS 2024

$$L(s) = \frac{2(s+5)(s-0.1)}{s(s+1)(s+10)}$$



What is the number of unstable poles Z of the CL system for $0.25 < -1/k < 0.75$?

A) Z = 0

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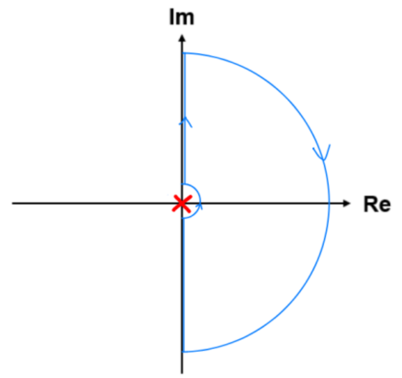
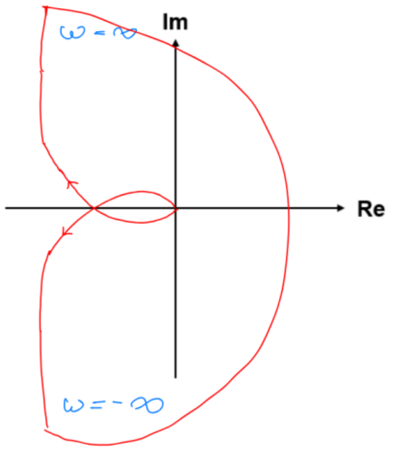
Special Case

What happens when there are poles on the Im – axis of the open loop system? Do they count into the Contour or not?
For that we will just follow a special rule:

When excluding the poles, going around them CCW, close the Nyquist plot at infinity clockwise.
For every pole excluded, add $+180°$ when closing the Nyquist



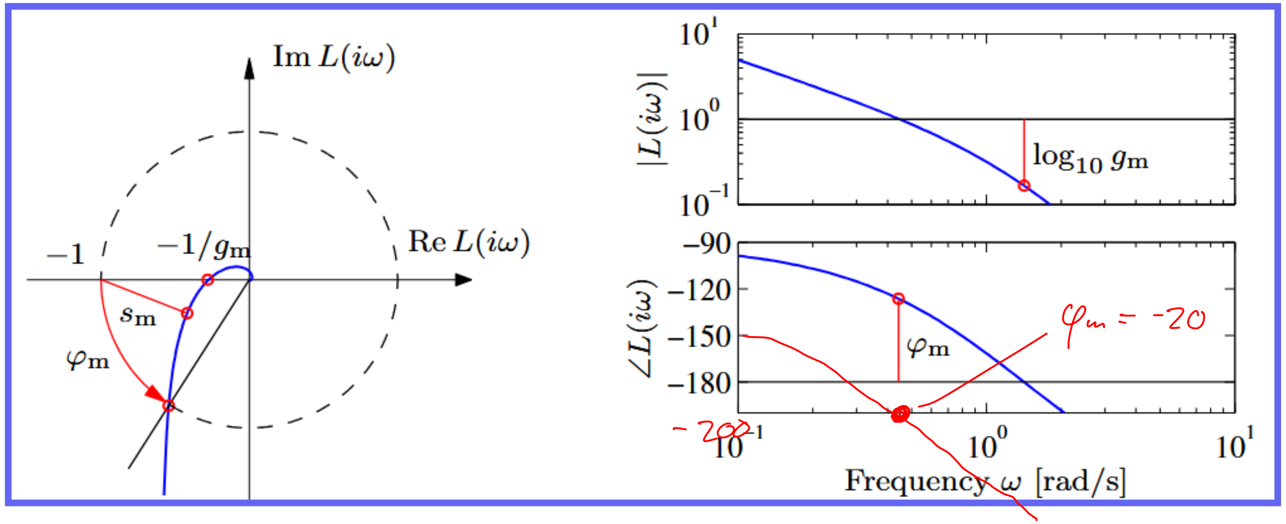


Stability Margins (also important for today)

We can determine how close our open-loop system is to being unstable. For that the following thing needs to be fullfilled: $|L(j\omega)| < 1$ **whenever** $\angle L(j\omega) = \pm 180°$.

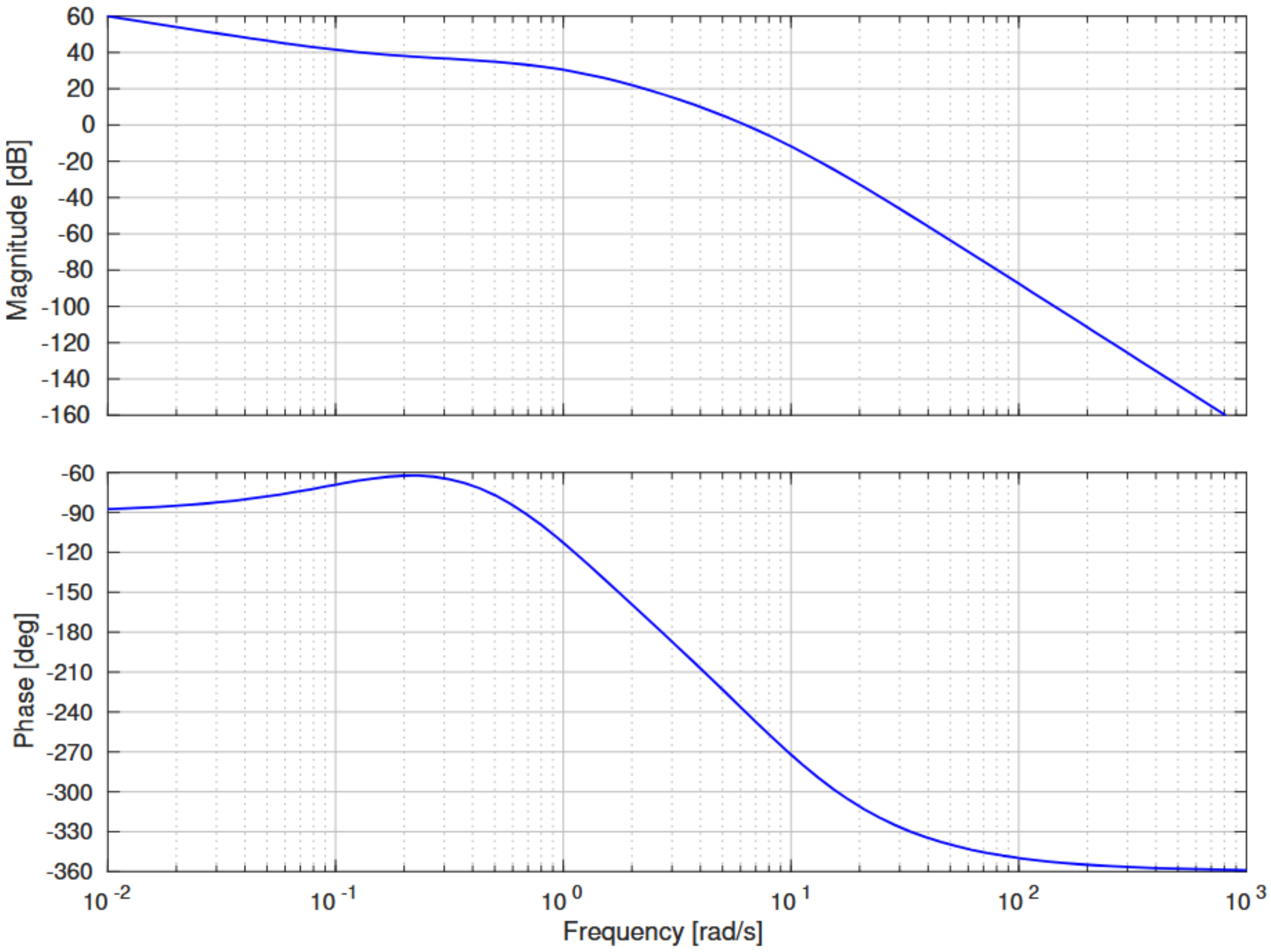
Gain Margin GM / $g_{m,dB}$: It indicates how much we can increase the magnitude at the phase crossover frequency ω_{pc}, at which the phase crosses $\pm 180°$, before encircling -1

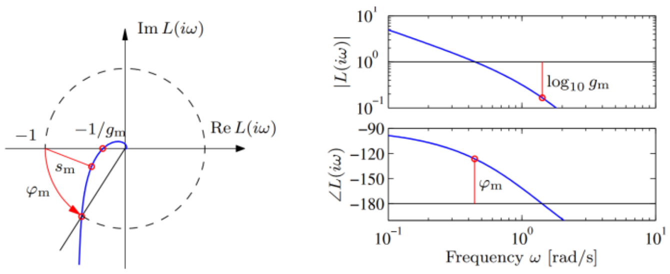
Phase Margin PM / φ_m: It indicates much much we can change the phase at the crossover frequency ω_c, at which the magnitude is 1 (0 in dB), before encircling -1



Phase margin:	Gain margin:
Find **cross-over frequency** ω_c: $\lvert L(j\omega_c)\rvert = 1$ $\lvert L(j\omega_c)\rvert_{dB} = 0$ Find the phase $\varphi = \angle L(j\omega_c)$ and set the phase margin: $\varphi_m = \varphi + 180°$	Find the frequency ω at which $\angle L(j\omega) = -180°$. Then find $\lvert L(j\omega)\rvert_{dB}$ for the same ω Compute the gain margin: $g_{m,dB} = 0 - \lvert L(j\omega)\rvert_{dB}$ $g_m = 10^{\frac{g_{m,dB}}{20}}$

FS 2024





Phase margin:

Find **cross-over frequency** ω_c:

$$|L(j\omega_c)| = 1$$
$$|L(j\omega_c)|_{dB} = 0$$

Find the phase $\varphi = \angle L(j\omega_c)$ and set the phase margin:

$$\varphi_m = \varphi + 180°$$

Gain margin:

Find the frequency ω at which $\angle L(j\omega) = -180°$. Then find $|L(j\omega)|_{dB}$ for the same ω

Compute the gain margin:

$$g_{m,dB} = 0 - |L(j\omega)|_{dB}$$
$$g_m = 10^{\frac{g_{m,dB}}{20}}$$

The PM is best described by:

A) $PM \approx 58°$

B) $PM \approx -61°$

C) $PM \approx 34°$

D) $PM \approx -26°$

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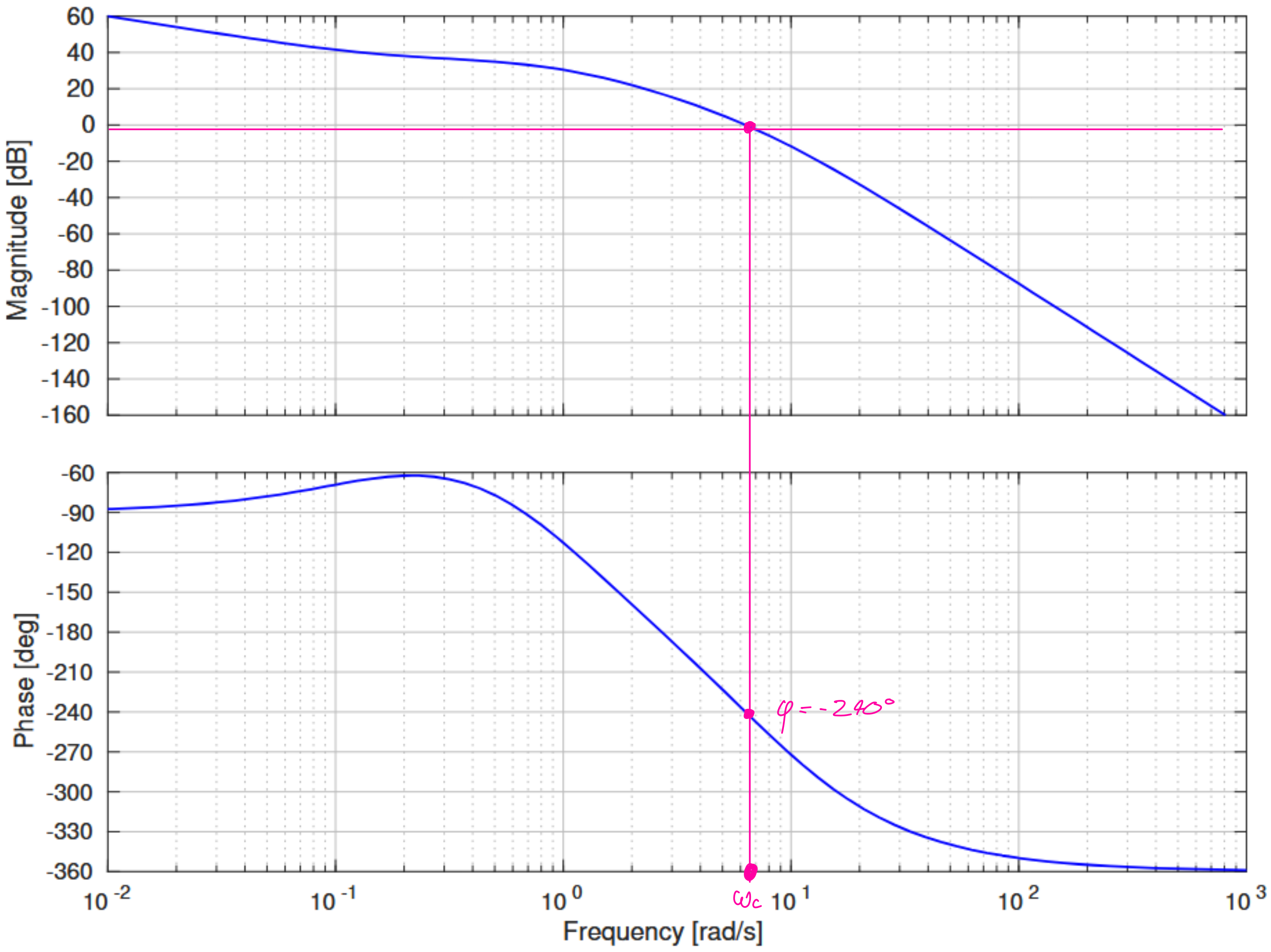
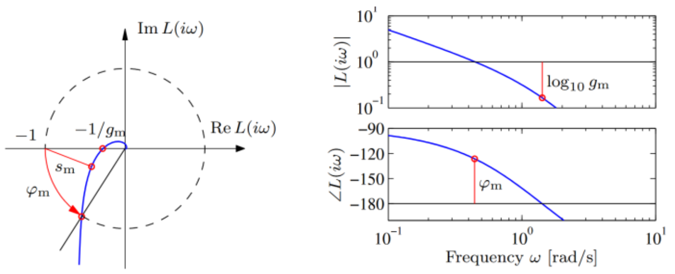


$-240 + 180$

$\approx -60°$



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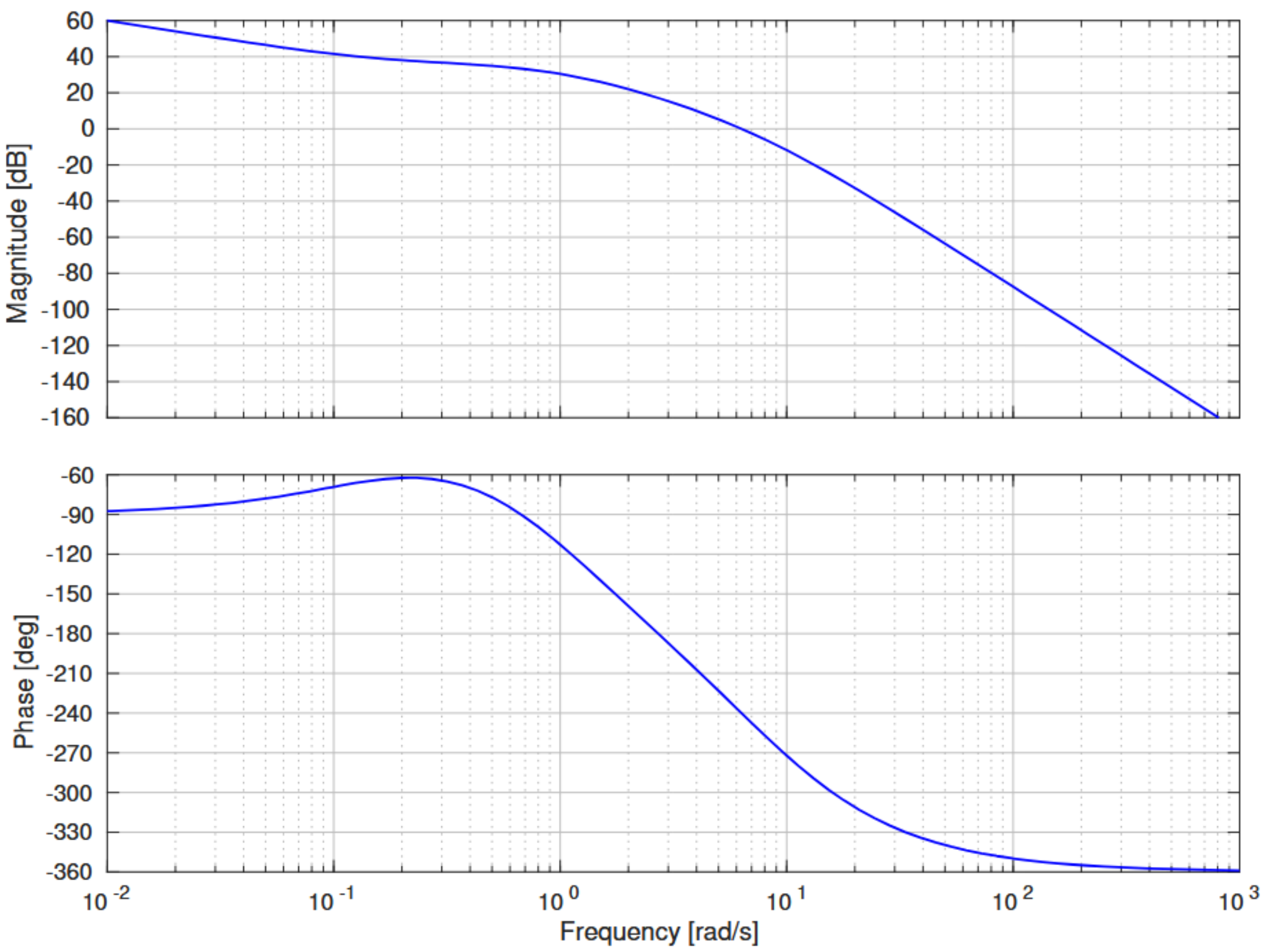

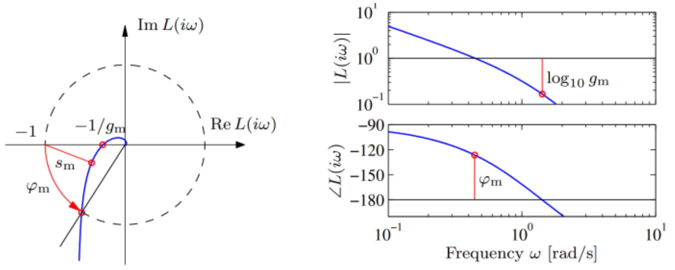

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The GM is best described by:

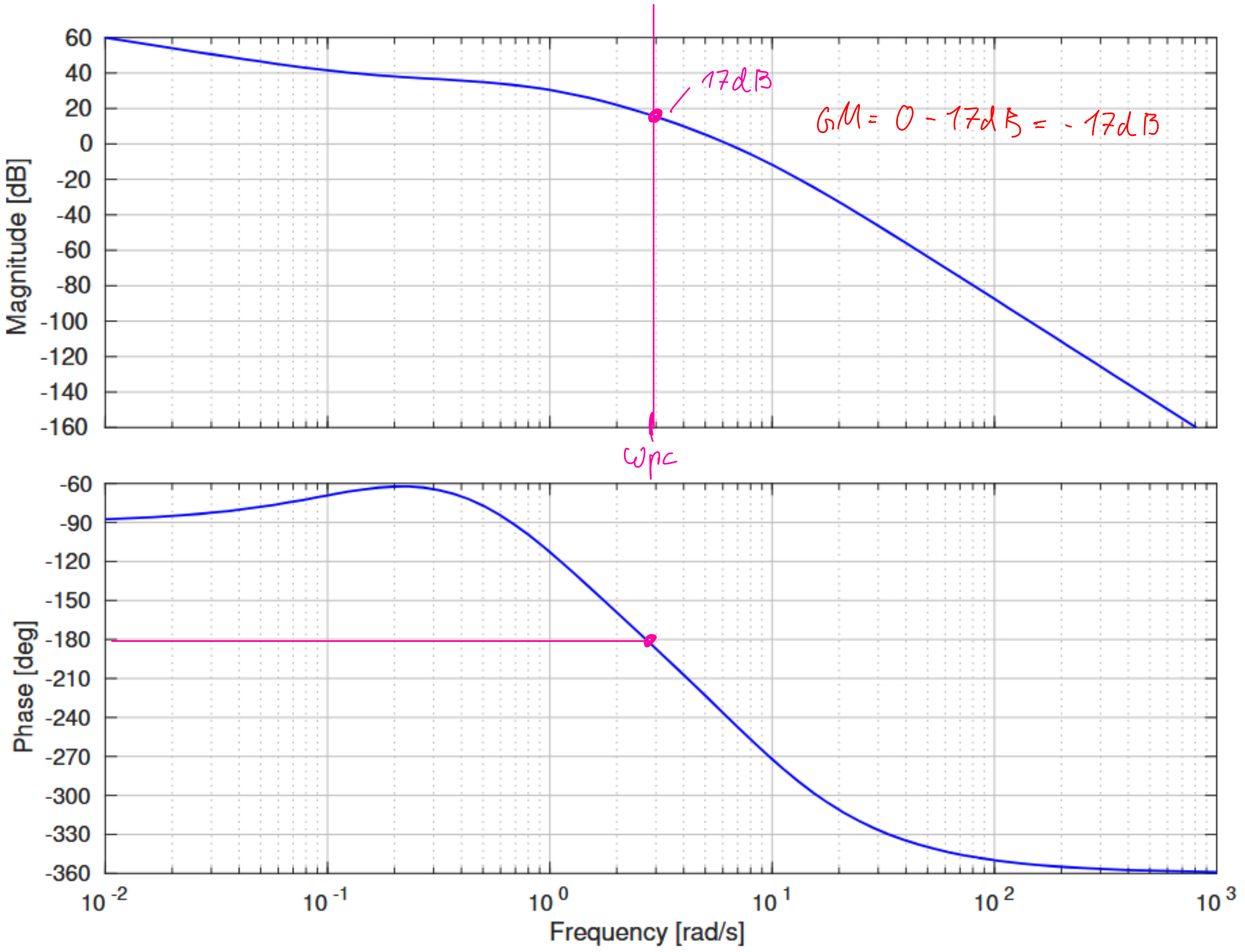
A) $GM \approx 18$ dB

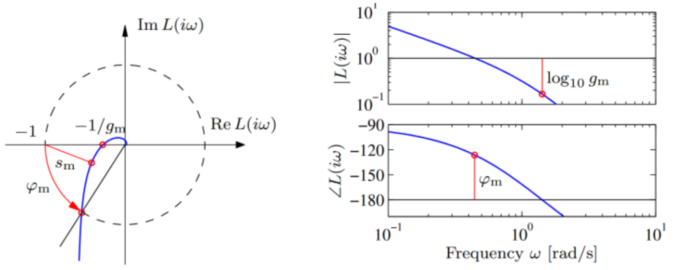
B) $GM \approx 5$ dB

C) $GM \approx 90$ dB

D) $GM \approx -17$ dB

FS 2024





Phase margin:	Gain margin:
Find **cross-over frequency** ω_c:	Find the frequency ω at which $\angle L(j\omega) = -180°$. Then find $\lvert L(j\omega)\rvert_{dB}$ for the same ω
$\lvert L(j\omega_c)\rvert = 1$ $\lvert L(j\omega_c)\rvert_{dB} = 0$	Compute the gain margin:
Find the phase $\varphi = \angle L(j\omega_c)$ and set the phase margin:	$g_{m,dB} = 0 - \lvert L(j\omega)\rvert_{dB}$ $g_m = 10^{\frac{g_{m,dB}}{20}}$
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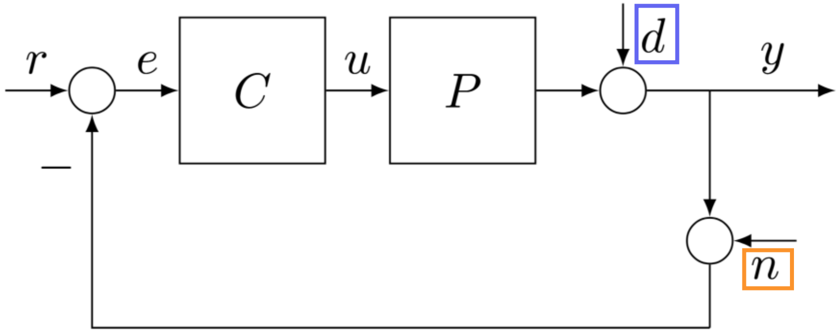
Theory Recap

Frequency Domain Specifications

Closed Loop Transfer Function



Open Loop TF

$$L(s) = C(s)P(s)$$

maps: e ➞ y

Closed Loop TF (Complementary Sensitivity)

$$T(s) = \frac{L(s)}{1+L(s)}$$

maps: r ➞ y, n ➞ y

Sensitivity Function

$$S(s) = \frac{1}{1+L(s)}$$

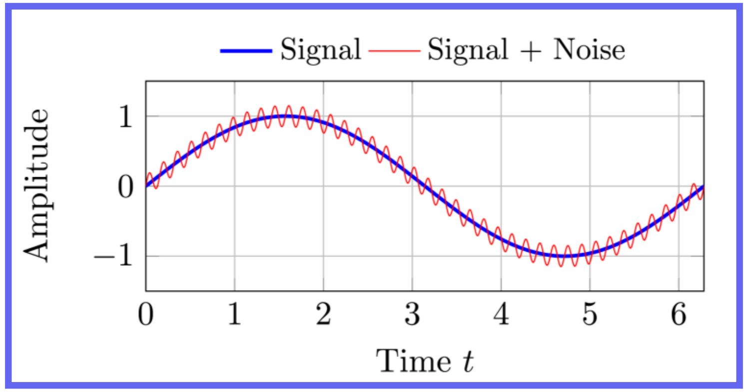
maps: r ➞ e, d ➞ y

System Goals

Disturbance Rejection describes how effectively the system maintains performance despite external disturbances $d(t)$. Disturbances are usually a **low frequency** phenomenon and can be mapped to the output by $\boldsymbol{S}(\boldsymbol{s})$

Command Tracking tells us how accurately the system output follows a reference signal $r(t)$.

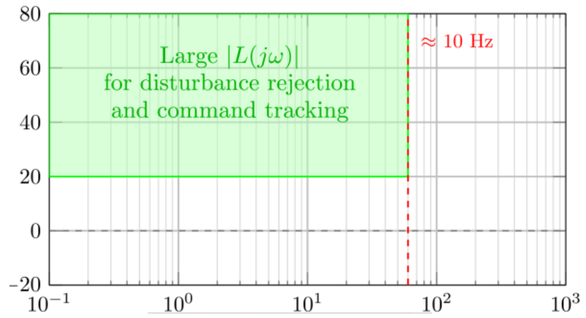
Noise Rejection tells us how well we can attenuate (abschwächen) noise $n(t)$ that is present in the CL-System. Noise is usually a **high frequency** phenomenon and can be mapped to the output by $\boldsymbol{T}(\boldsymbol{s})$.



Specifications

Since $S(s)$ maps the **disturbance signal $d(t)$ to the output $y(t)$,** we want $S(s)$ to be small for frequencies in which disturbances are present, therefore for low frequencies.

$$|S(j\omega)| \ll 1 \quad \text{for low } \omega \qquad |S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right| \quad \Rightarrow \quad |L(j\omega)| \gg 1$$

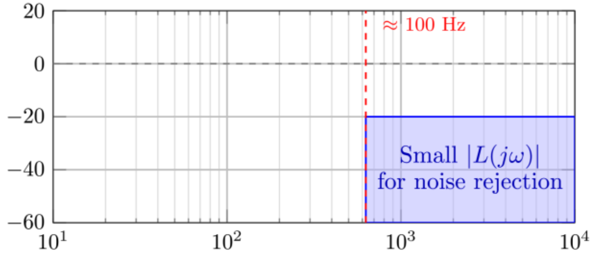


Since T(s) maps the **noise signal n(t) to the output $y(t)$,** we want T(s) to be small for frequencies in which noise is present, therefore for high frequencies.

$$|T(j\omega)| \ll 1 \quad \text{for high } \omega \qquad |T(s)| = \left|\frac{L(s)}{1+L(s)}\right| \quad \Rightarrow \quad |L(j\omega)| \ll 1$$

≈ 1 , $\frac{L(s)}{1} \approx L(s)$

$$|T(s)| \approx |L(s)|$$



Tradeoff (Waterbed Effect)

$$S(s) + T(s) = \frac{1}{1 + L(s)} + \frac{L(s)}{1 + L(s)} = \frac{1 + L(s)}{1 + L(s)} = 1$$

We can see, that when summing the two functions, **they add up to 1**, meaning when one is small, the other one is big. So in the end we have a trade off at a given frequency between these two.

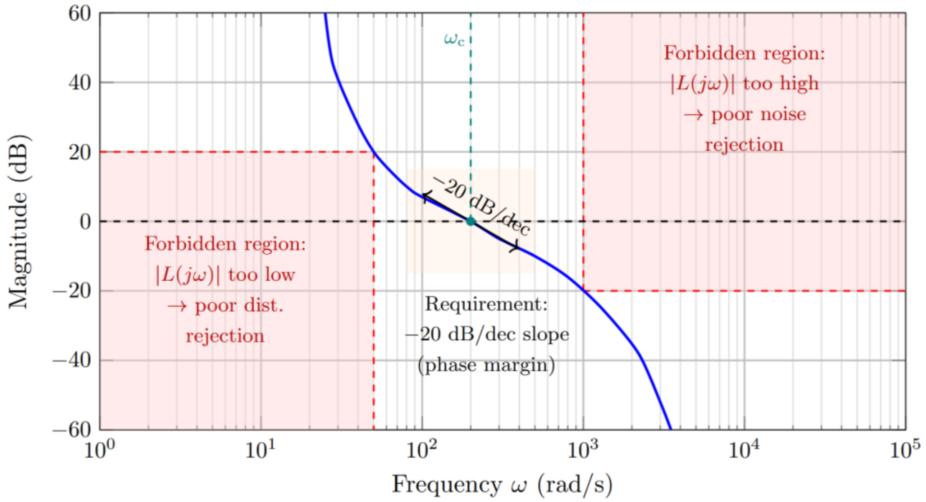
However, an important observastion that helps us solve this dilemma, is that **disturbances** are present at **low frequencies** (up to $10Hz$), and **noise** is present at **high frequencies** (more than $100Hz$).

$$|L(j\omega)| \gg 1 \quad \text{for low } \omega,$$

$$|L(j\omega)| \ll 1 \quad \text{for high } \omega$$

Bode Obstacle Course

Bode Obstacle Course



We will want to make our bode plot avoid certain regionsand have certain specifications.
This is typically reffered to as the «Bode Obstacle Course».
We will later see how to actually do this by changing our controller

Bandwidth

The **Bandwidth (BW)** of a system tells us the maximum frequency for which our system can track a command with a factor of ≈ 0.7, which corresponds to

$$|T(j\omega)| > \frac{1}{\sqrt{2}} \quad (\approx -3\text{dB})$$

It can be interpreted as how fast the closed-loop can react to changing references.

A large bandwidth means that high frequencies can easily pass

A small badnwidth means that even low frequencies experience attenuation

Very Important

Hint: Often the Bandwidth is asked in Hz, then use $f = \frac{\omega}{2\pi}$

Bandwidth

By setting the $|T(j\omega_{\text{bw}})| = \frac{1}{\sqrt{2}}$ and remembering that $|T(j\omega_{\text{bw}})| = \frac{1}{\sqrt{2}} = \frac{|L(j\omega_{\text{bw}})|}{|1+L(j\omega_{\text{bw}})|}$ $= \frac{|j|}{|1+j|}$

We find one solution where $L(j\omega_{\text{bw}}) = j$ and $|L(j\omega_{\text{bw}})| = 1$

Now remember that $|L(j\omega)| = 1 = [0]_{\text{dB}}$ is true for the crossover frequency ω_c.

Therefore we can assume that $\omega_{\text{bw}} \approx \omega_c$

Meaning, our open loop crossover frequency is equal to our closed loop bandwidth

For this assumption to hold, we will always want a slope of -20db/dec at the crossover frequency

Control Synthesis via Loop Shaping

We saw how we can tune the feedback gain k with the root locus to achieve desired CL behaviour.
This approach however is limited, since we can only tune a single parameter at a time.

Today we will learn about 2 methods to design a controller in the frequency domain:

1. **Brute Force Approach**

2. **Loop Shaping Approach**

Brute Force Approach

Brute Force Approach

1. We start by designing an appropriate open loop TF $L_{des}(s)$ that fulfills all our desires.

2. We remember that we can write $L_{des}(s) = C(s)P(s)$

3. Now we can find our controller C with $C(s) = \frac{L_{des}(s)}{P(s)}$ where P is our plant.

This might seem easy to do, but:

- The open loop must be stable and minimum-phase. If not, instabilities will just get cxancelled

- The compensator / controller may be unnecessarily complicated

- It may be that the compensator is not causal

Loop Shaping Approach

Loop Shaping Approach

1. Rely on standard control elements with known characteristics and merge them

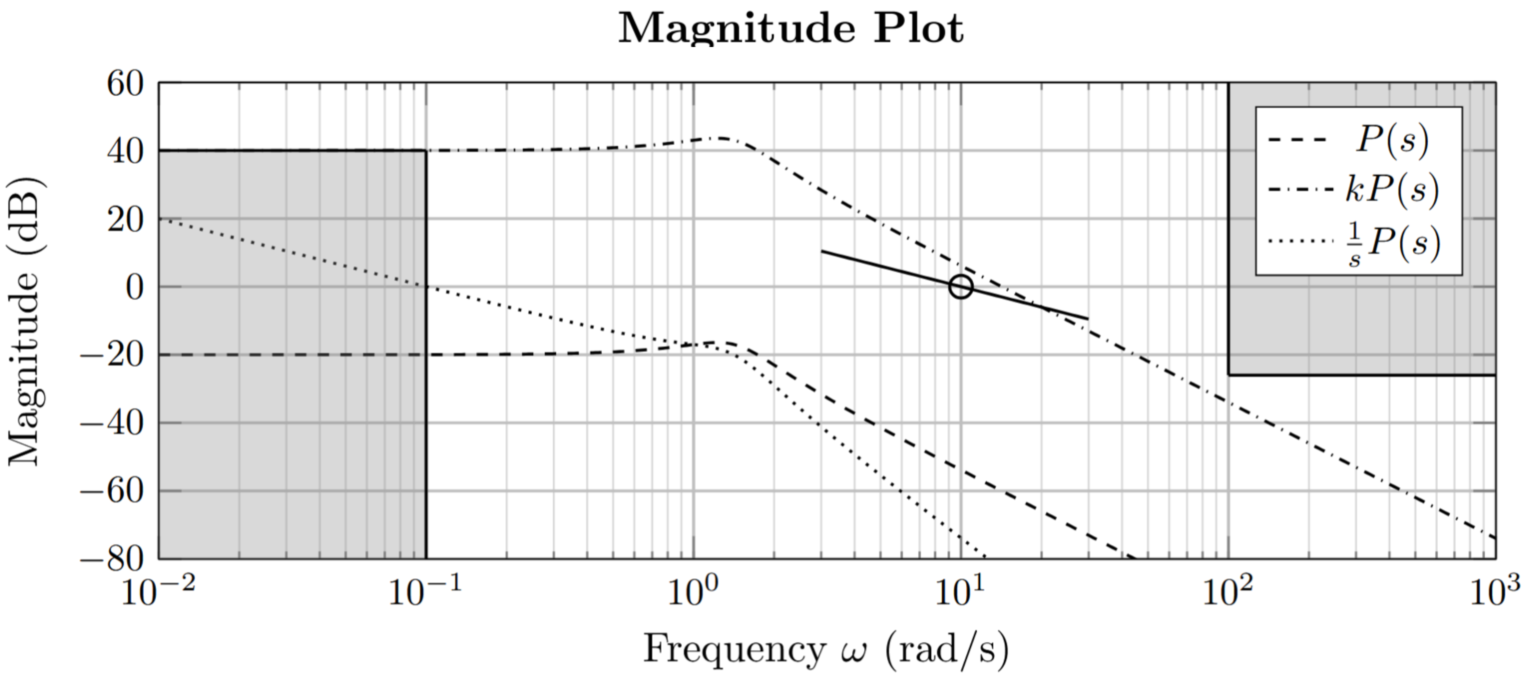
$$C(s) = C_1(s) \cdot C_2(s) \cdot \ldots \cdot C_n(s)$$

The most basic (but super useful) ones are:

Known
- Proportional gain: k,
- Integrator: $\frac{1}{s}$,

New
- Lead element: $\frac{s/a+1}{s/b+1}$, with $0 < a < b$
- Lag element: $\frac{s/a+1}{s/b+1}$, with $0 < b < a$.

Known Elements



For Frequency Response:

- A proportional gain **shifts the magnitude** over all frequencies

- An integrator adds a **slope of $-20\ dB\ /\ dec$** over all frequencies and shifts te **phase plot down by $-90°$**

New Elements

The elements we are going to investigate are **Lead- and Lag-Compensators.**

By **introducing a pole and a zero** at strategically chosen locations, we can manipulate the magnitude and phase response of the system systematically

Lead Compensator

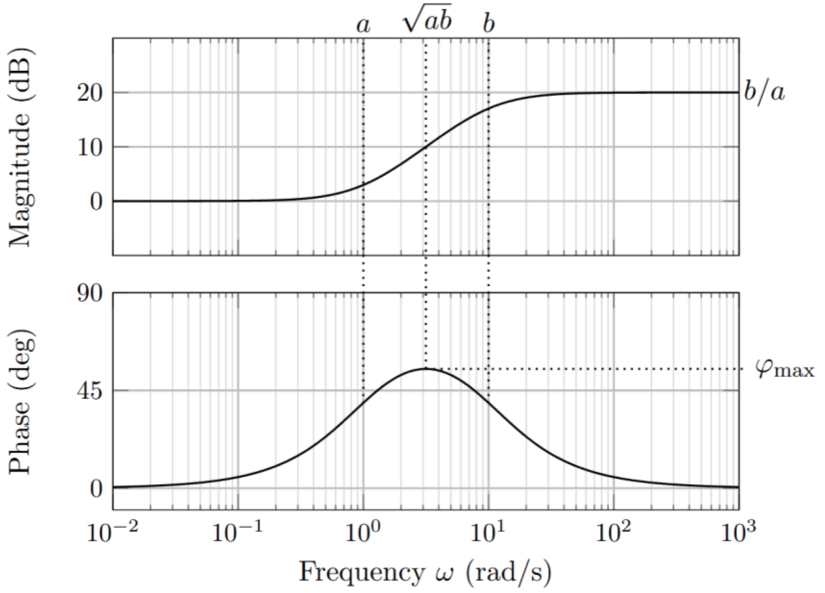
$$C_{\text{lead}}(s) = \frac{s/a+1}{s/b+1}, \quad 0 < a < b$$

Minimum Phase Zero before Stable Pole

- **Magnitude Response:** The magnitude at high frequencies is increased by $\frac{b}{a}$. Low frequencies are not affected. The slope is increased by $+20\text{dB/dec}$ between a and b
- **Phase Response**: The phase is increased around $\sqrt{ab}$ by up $to 90°$. The actual maximum increase is found by

$$\varphi_{\max} = 2\arctan\left(\sqrt{\frac{b}{a}}\right) - 90°$$

Bode Plot of Lead Element



$$a = 1\,\text{rad/s and } b = 10\,\text{rad/s}$$

$$\frac{b}{a} = 10 \implies 20\text{dB}$$

$$2\arctan\left(\sqrt{10}\right) - 90° \approx 54.9°$$

$$\text{around } \omega = \sqrt{10} \approx 3.16\,\text{rad/s}$$

Lead Compensator

Typically we use lead compensators in combination with proportional controll to **increase the phase margin** at a specifiable crossover frequency ω_c. To achieve this, we follow the steps below:

1. Pick $\sqrt{ab}$ at the desired crossover frequency ω_c,
2. Pick b/a such that the desired phase increase is achieved,
3. Adjust k, such to set the crossover frequency to the desired frequency

Since this increases magnitude at high frequencies, it may lead to less noise rejection

Lag Compensator

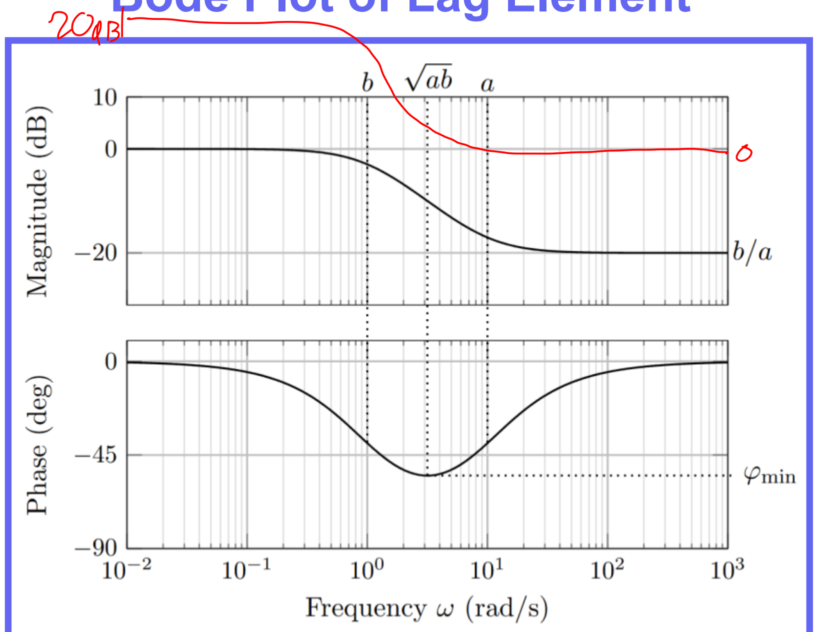
$$C_{\text{lag}}(s) = \frac{s/a+1}{s/b+1}, \quad 0 < b < a$$

Stable Pole before Minimum Phase Zero

- **Magnitude Response:** The magnitude at high frequencies is decreased by $\frac{b}{a}$. Low frequencies are not affected. The slope is **decreased** by 20dB/dec between b and a
- **Phase Response**: The phase is decreased around $\sqrt{ab}$ by up $to 90°$. The actual maximum ~~increase~~ decrease is found by

$$\varphi_{\min} = 2\arctan\left(\sqrt{\frac{b}{a}}\right) - 90°$$

Bode Plot of Lag Element



$$b = 1,\ a = 10$$

$$\frac{b}{a} = 0.1 \implies -20\text{dB}$$

$$2\arctan\left(\sqrt{0.1}\right) - 90° \approx -54.9°$$

around $\omega = \sqrt{10} \approx 3.16\,\text{rad/s}$

Lag Compensator

Typically we use lag compensators for **disturbance rejection and command tracking.** We found, that $L(s)$ should have high magnitude at low frequencies. With an additional proportional gain, we follow the steps below.

1. Pick a/b as the desired increase in magnitude at low frequencies,
2. Multiply the proportional gain k by a/b. This ensures that the response at high frequencies is unaffected and increases the magnitude at low frequencies.
3. Pick a sufficiently smaller than the crossover frequency such that phase margins are not affected too much.

This may also introduce phase lag at low frequencies and potentially reduce phase margins.

Loop Shaping

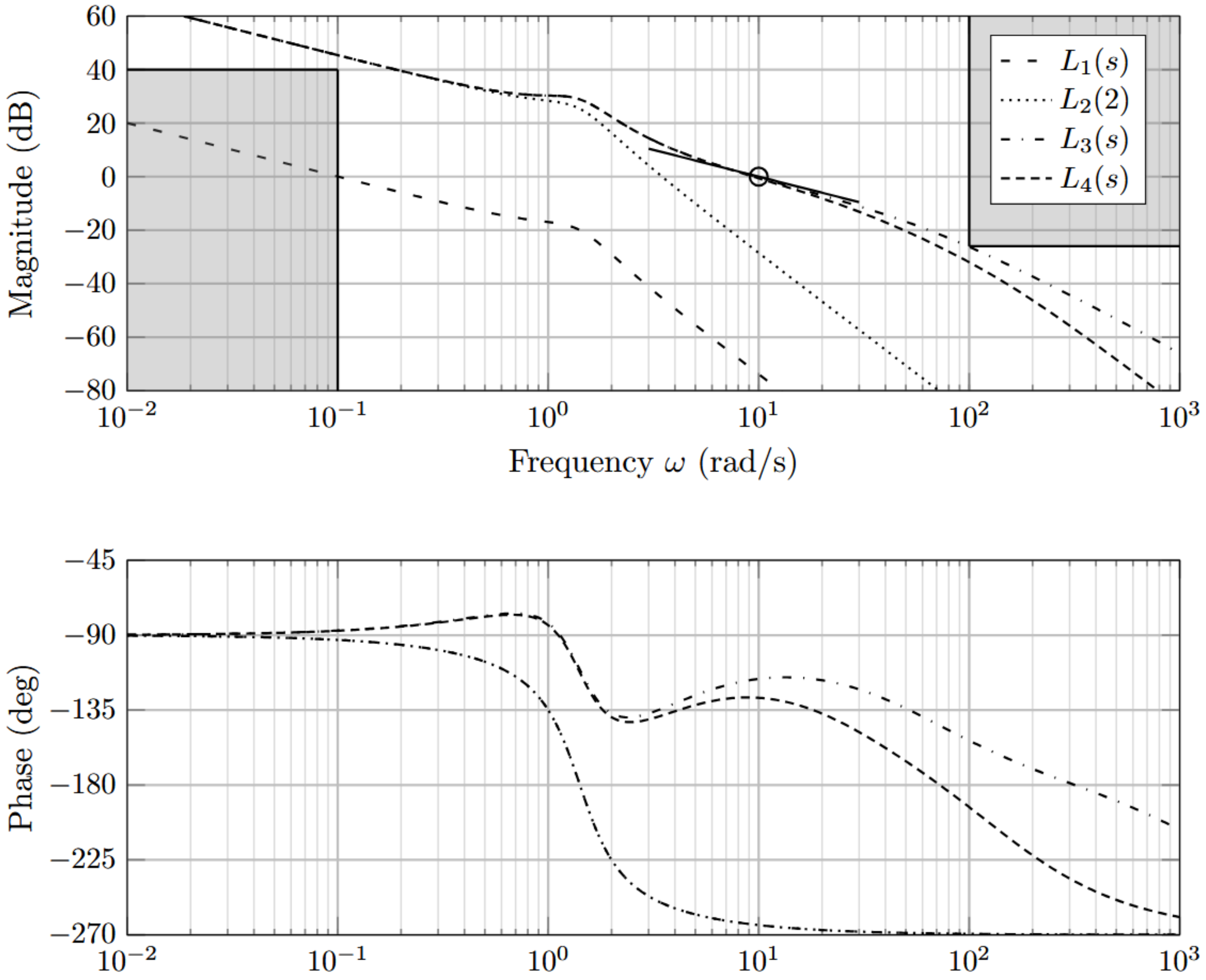
$$P(s) = \frac{0.2}{s^2 + 0.2s + 0.2}$$

$$C_1(s) = \frac{1}{s}$$

$$C_2(s) = 185\frac{1}{s}$$

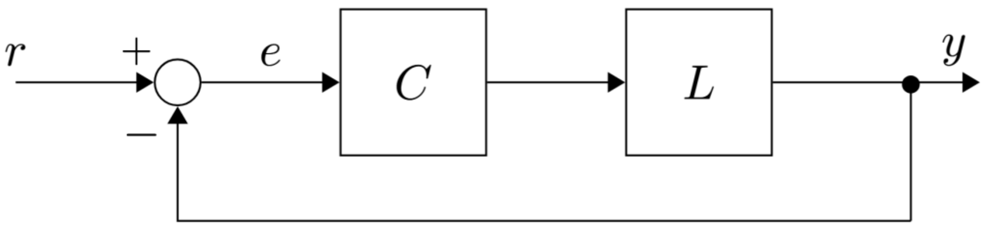
$$C_3(s) = 185\frac{1}{s}\left(\frac{s/2+1}{s/120+1}\right)^2$$

$$C_4(s) = 185\frac{1}{s}\frac{s/80+1}{s/30+1}\left(\frac{s/2+1}{s/120+1}\right)^2$$



FS 2024

$$C(s) = \frac{s/a + 1}{s/b + 1}, \quad a, b \in \mathbb{R}$$



$$\Delta\varphi \approx 2 \cdot \arctan(\sqrt{b/a}) - 90°$$

1. Choose a/b as the desired increase in magnitude at low ω
2. Pick $\sqrt{a \cdot b}$ as far as possible from the desired crossover frequency ω_c to not risk instability

What choice of the parameters a und b satisfy the goal to reduce the phase by $30°$ at $\omega = 50$ rad/s

A) $a = \frac{50}{\sqrt{3}}, b = 50\sqrt{3}$

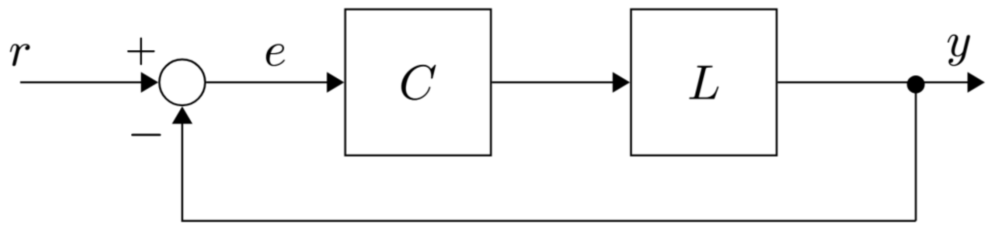
C) $a = 50\sqrt{3}, b = \frac{50}{\sqrt{3}}$

B) $a = \frac{5}{\sqrt{3}}, b = 500\sqrt{3}$

D) $a = \frac{500}{\sqrt{3}}, b = 5\sqrt{3}$

FS 2024

$$C(s) = \frac{s/a + 1}{s/b + 1}, \quad a, b \in \mathbb{R}$$



$\Delta\varphi \approx 2 \cdot \arctan(\sqrt{b/a}) - 90°$

1. Choose a/b as the desired increase in magnitude at low ω
2. Pick $\sqrt{a \cdot b}$ as far as possible from the desired crossover frequency ω_c to not risk instability

$\varphi_{min} = -30° = 2 \cdot \arctan(\sqrt{b/a}) - 90$

$30° = \arctan(\sqrt{b/a})$

$\tan(30°) = \sqrt{b/a} = \frac{1}{\sqrt{3}}$

$\frac{b}{a} = \frac{1}{3}$

$b = \frac{1}{3} a \rightarrow \ldots\ldots$

What choice of the parameters a und b satisfy the goal to reduce the phase by $30°$ at $\omega = 50$ rad/s

A) $a = \frac{50}{\sqrt{3}}, b = 50\sqrt{3}$

B) $a = \frac{5}{\sqrt{3}}, b = 500\sqrt{3}$

C) $a = 50\sqrt{3}, b = \frac{50}{\sqrt{3}}$

D) $a = \frac{500}{\sqrt{3}}, b = 5\sqrt{3}$

$\frac{50/\sqrt{3}}{50\sqrt{3}} = \frac{1}{3}$

$\sqrt{50\sqrt{3} \cdot \frac{50}{\sqrt{3}}} = \sqrt{50^2} = 50$

$\frac{5\sqrt{3}}{500/\sqrt{3}} = 3$

HS 2023

A A lead compensator is typically used to increase the low frequency gain, and thus improve the steady-state error.
B A lead compensator can increase the closed-loop noise sensitivity at high frequencies.
C A lag compensator is typically used to make the system more robust by increasing the phase margin.

Mark all the correct statements

A)

B)

C)

HS 2023

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Mark all the correct statements

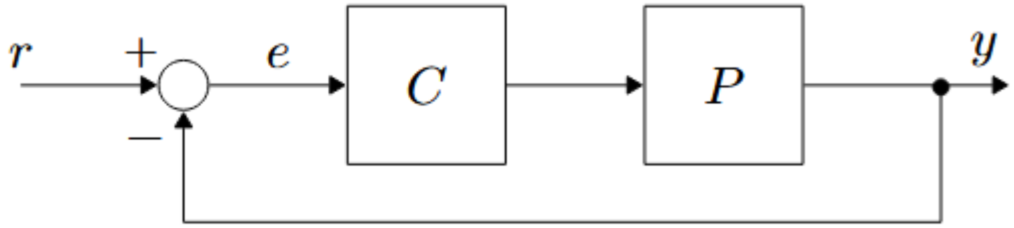
A)

B)

C)

FS 2023

$$P(s) = \frac{1}{s+1}, \text{ and } C(s) = \frac{k+s}{s}$$



What k makes the gain crossover frequency be at $\omega_{gc} = 10$ rad/s ?

A) $400\pi^2$

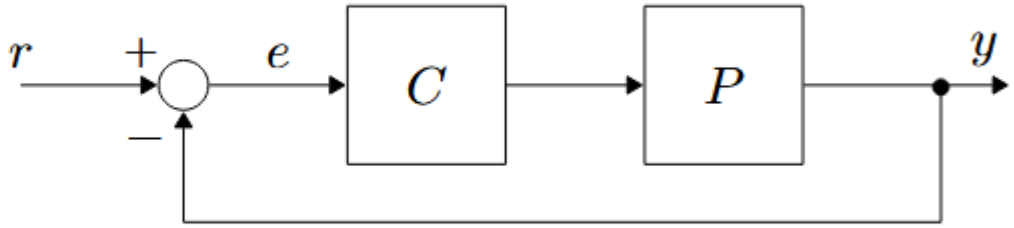
C) $\frac{\pi}{20}$

B) 100

D) 0.25

FS 2023

$$P(s) = \frac{1}{s+1}, \text{ and } C(s) = \frac{k+s}{s}$$



What k makes the gain crossover frequency be at $\omega_{gc} = 10$ rad/s ?

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$$P(s)C(s) = \frac{1}{s+1} \cdot \frac{k+s}{s} = \frac{k+s}{s^2+s}$$

$$L(j\omega_c) = \frac{k+10j}{-100+10j}$$

$$|A/B| = 1 \implies |A| = |B|$$

Q&A Session / Done

Feedback

jschultev.github.io/personal_website/Feedback

HS 2023

For you to look at if you want. Won't go through it now. However it is nice!

- A: Assume k_I and k_D fixed. Increasing k_P can make the system respond faster.
- B: Assume k_D and k_P fixed. In order to guarantee a zero steady-state error $e_{ss} = 0$ to step references, it is necessary that $k_I \neq 0$.
- C: Assume k_I and k_P fixed. By tuning k_D we can reduce the steady-state error e_{ss} to step references to zero.
- D: Assume k_I and k_P fixed. By increasing k_D we can reduce the overshoot of the closed-loop step response.

Mark all the correct statements

A) | C)

B) | D)

HS 2023

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Mark all the correct statements

A) C)

B) D)